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Green's functions, temperature and heat flux in the rectangle

Kevin D. Cole [a,*], David H.Y. Yen [b]

[a] Mechanical Engineering Department, University of Nebraska, N123 Walter Scott Engr. Center, Lincoln, NE 68588-0656, USA

[b] Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

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Abstract

Steady heat conduction in the rectangle is treated with the method of Green's functions. Single-sum series for the Green's functions are reported in terms of exponentials which have better numerical properties than hyperbolic functions. Series expressions for temperature and heat flux caused by spatially uniform effects are presented. The numerical convergence of these series is improved, in some cases by a factor of 1000, by replacing slowly converging portions of the series with fully summed forms. This work is motivated by high-accuracy verification of finite-difference and finite-element codes.

Keywords: Green's functions; Laplace equation; Rectangle; Series convergence

1. Introduction

The method of Green's functions (GF) applies to linear differential equations that describe a wide variety of physical phenomena, including heat conduction, fluid flow, and electrochemical potential. In this method the boundary value problem for the temperature is restated into an integral expression that involves the known boundary conditions and the GF. If the GF is known and if the integral expressions can be evaluated then the method of GF is a powerful tool for solving many different problems. This paper applies the method of GF to evaluate steady-state temperature and heat flux in the rectangle. One motivation for the work is high-accuracy verification of finite-difference and finite-element codes.

The pertinent literature is summarized next. Several books give a good overview of the GF method such as Morse and Feshbach [1], Carlsaw and Jaeger [2] and Stakgold [3]. Barton [4] carefully discusses the properties of the Dirac delta function and describes the pseudo GF for the Neumann boundary condition. Two books by Butkovskii [5,6] contain many GF organized according to the type of differential equation. The differential equations are categorized according to a number system for the number of spatial dimensions, the order of the highest time derivative, and the order of the highest spatial derivative. Although Butkovskii's number system clearly distinguishes different equations, there are no subdivisions for the various coordinate systems and boundary conditions. Beck et al. [7] give extensive tables of GF for heat conduction and diffusion. The GF are organized with a number system for the number of spatial dimensions, the type of coordinate system, and the type of boundary conditions. Most of the book is devoted to transient heat conduction and few 2D steady GF are given.

Dolgova and Melnikov [8] discuss the steady 2D heat conduction in Cartesian and cylindrical coordinates. Fourier series expansions along one coordinate direction are used to produce single-sum series for the GF. Three examples of GF for the rectangle are given. Most importantly, the slowly converging portions of the series for the GF are identified and replaced with closed-form expressions. This approach has been extended and expanded in two recent books by Melnikov [9,10] to improve the numerical convergence of GF for a variety of equations, coordinate systems, and geometries. The chapters on Laplace and Helmholtz equations include sections on the rectangle, and several GF are given.

* Corresponding author. Tel.: +1-402-472-5857.
E-mail address: kcole1@unl.edu (K.D. Cole).

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Although wide ranging, the improvement of convergence is applied only to GF.

Marshall [11] discusses Laplace equation solutions for a rectangle with Neumann boundary conditions applied to electrochemical cells. Similar to Melnikov, Marshall replaces the slowly converging portions of the GF with closed-form expressions, some of which are constructed from 1D GF. Numerical examples are given for specified-flux boundaries that represent two small electrodes embedded on opposite sides of the rectangle.

Previous work on steady heat conduction in the rectangle by the first author [12] provides a complete list of all single-sum GF with boundary conditions of type 1, 2, or 3. The GF are organized and identified by Beck's numbering system.

The contribution of the present paper is threefold. First, GF for the rectangle are given in single-sum form involving exponentials which are numerically better behaved than the hyperbolic functions previously published. Second, integrals are carried out to produce series expressions for temperature and heat flux caused by spatially uniform boundary conditions and spatially uniform volume energy generation. Third, the numerical properties of the series expressions for temperature and heat flux are improved by replacing slowly converging portions of the series with fully summed forms. The work encompasses 81 rectangular geometries containing any combination of boundaries of types 1, 2, and 3.

2. Temperature problem

Consider the steady temperature in the rectangle caused either by heating at the boundary or by internal energy generation. The temperature satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{g(x,y)}{k} \quad 0 < x < L,\ 0 < y < W,$$
$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i \quad \text{for boundaries } i = 1, 2, 3, \text{ or } 4. \tag{1}$$

Here n_i is the outward normal on each face of the rectangle. The boundary condition represents one of three types at each surface: type 1 for $k_i = 0$, $h_i = 1$, and f_i a specified temperature; type 2 for $k_i = k$, $h_i = 0$, and f_i a specified heat flux; and type 3 for $k_i = k$ and $f_i = h_i T_\infty$ for convection to surroundings at temperature T_∞. Heat transfer coefficient h_i must be uniform on the ith boundary.

The temperature can be stated in the form of integrals with the method of Green's functions. If Green's function G is known, then the temperature that satisfies Eq. (1) is given by

$$T(x,y) = \int_{x'=0}^{L} \int_{y'=0}^{W} \frac{g(x',y')}{k} G(x,y \mid x',y')\,dx'\,dy'$$

(for volume energy generation)

$$+ \sum_{j=1}^{4} \int_{s_j} \frac{f_j}{k} G(x,y \mid x'_j, y'_j)\,ds'_j$$

(for boundary conditions of types 2 and 3 [1])

$$- \sum_{i=1}^{4} \int_{s_i} f_i \frac{\partial G(x,y \mid x'_i, y'_i)}{\partial n'_i}\,ds'_i \tag{2}$$

(for boundary conditions of type 1 only).

The same Green's function appears in each integral but is evaluated at locations appropriate for each integral. Here position (x'_i, y'_i) is located on surface s_i and n_i is the outward-facing unit normal on this surface. The two summations represent all possible combinations of boundary conditions, but with only one type of boundary condition on each of four surfaces of the rectangle. Mixed-type boundary conditions are not treated.

3. Definition of the GF

The steady Green's function represents the response at point (x,y) caused by a point source of heat located at (x',y'). The GF associated with Eq. (1) is given by

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x-x')\delta(y-y')$$
$$0 < x < L,\ 0 < y < W, \tag{3}$$
$$k_i \frac{\partial G}{\partial n_i} + h_i G = 0 \quad \text{for faces } i = 1,2,3,4.$$

The homogeneous boundary conditions are of the same type as the temperature problem, but the volume energy generation is replaced by a Dirac delta function, δ. Most of the quantities in this discussion have units: (x,y) are each (m); $\delta = (\mathrm{m}^{-1})$; $k = (\mathrm{W/m/K})$; $h = (\mathrm{W/m^2/K})$; and function G is dimensionless for steady heat conduction in the rectangle.

4. GF number

The specific GF and the specific geometry are identified by a "number" of the form X*IJ*Y*KL* in which X and Y represent the coordinate axes, and the letters following each axis name take on values 1, 2, or 3 to represent the type of boundary conditions present at the body faces normal to that axis. For example, number X12 represents boundary conditions of type 1 at $x = 0$ and type 2 at $x = L$. As another example, number

[1] A second form of this integral is discussed in Section 6.4 for type 3 boundaries.

X11Y13 describes a GF for a rectangle with three faces with type 1 boundaries ($G = 0$) and the face at $y = W$ has a type 3 boundary (convection). See [7, Chapter 2], for additional details of the number system.

5. Single-summation form of the GF

The GF stated below contains a summation with eigenfunction Y_n, norm $N_y^{1/2}$, and kernel function P_n, as follows:

$$G(x, y \mid x', y') = \frac{1}{W} P_0(x, x') + \sum_{n=1}^{\infty} \frac{Y_n(y') Y_n(y)}{N_y(\gamma_n)} P_n(x, x'). \tag{4}$$

The summation term is needed for every GF. The first term with kernel function P_0 is needed only when Y22 is part of the GF number (when zero is an eigenvalue). This term is discussed in detail later.

In Eq. (4) the kernel functions P_n are placed along the x-axis. An alternate GF may be constructed by placing the kernel functions along the y-axis. Both forms of the GF are different series expansions of one unique solution to Eq. (3).

5.1. Eigenfunctions

The y-direction eigenfunction satisfies the following ordinary differential equation:

$$Y_n''(y) + \gamma_n^2 Y_n(y) = 0, \tag{5}$$

where γ_n is the associated eigenvalue. (Strictly speaking, the eigenvalues are γ_n^2, which can be shown to be real and non-negative. Without ambiguity we take the non-negative square root of γ_n^2 and shall refer to them as the "associated eigenvalues" for brevity.) There are nine different eigenfunctions associated with the nine possible boundary condition combinations YKL ($K, L = 1, 2$, or 3). Eigenfunctions $Y_n(y)$ are composed of sines and cosines, and are given in many texts. Table 1 contains the eigenfunctions and norms and Table 2 contains the associated eigenconditions (and eigenvalues for simple cases). For case Y22 the eigenvalue may also take on the value zero which requires special care.

5.2. Kernel functions

The method for obtaining kernel functions P_n will be discussed next. To obtain functions P_n, substitute the series for G given by Eq. (4) into Eq. (3). Additionally the y-portion of the Dirac delta function is replaced with the following identity:

$$\delta(y - y') = \sum_{n=0}^{\infty} \frac{Y_n(y') Y_n(y)}{N_y(\gamma_n)}. \tag{6}$$

Table 1
Eigenfunctions and inverse norm[a]

Case	$Y_n(y)$	N_y^{-1}
Y11	$\sin(\gamma_n y)$	$2/W$
Y12	$\sin(\gamma_n y)$	$2/W$
Y13	$\sin(\gamma_n y)$	$2\phi_{2n}/W$
Y21	$\cos(\gamma_n y)$	$2/W$
Y22	$\cos(\gamma_n y)$, $\gamma_n \neq 0$ 1, $\gamma_n = 0$	$2/W$ for $\gamma_n \neq 0$ $1/W$ for $\gamma_n = 0$
Y23	$\cos(\gamma_n y)$	$2\phi_{2n}/W$
Y31	$\sin(\gamma_n(W - y))$	$2\phi_{1n}/W$
Y32	$\cos(\gamma_n(W - y))$	$2\phi_{1n}/W$
Y33	$\gamma_n W \cos(\gamma_n y) + (h_1 W/k) \sin(\gamma_n y)$	$2\Phi_n/W$

Note: $\phi_{\text{in}} = \left[(\gamma_n W)^2 + (h_i W/k)^2\right] \div \left[(\gamma_n W)^2 + (h_i W/k)^2 + h_i W/k\right]$
$\Phi_n = \phi_{2n} \div \left[(\gamma_n W)^2 + (h_1 W/k)^2 + (h_1 W/k)\phi_{2n}\right]$

[a] Index $n = 1, 2, \ldots$ for all cases except Y22 with $n = 0, 1, 2, \ldots$

Table 2
Eigencondition and eigenvalues for $Y_n(y)$[a]

Case	Eigencondition	Eigenvalues
Y11	$\sin(\gamma_n W) = 0$	$\frac{n\pi}{W}$, $n = 1, 2, \ldots$
Y12	$\cos(\gamma_n W) = 0$	$\frac{(2n-1)\pi}{2W}$, $n = 1, 2, \ldots$
Y13	$\gamma_n W \cot(\gamma_n W) = -h_2 W/k$	
Y21	$\cos(\gamma_n W) = 0$	$\frac{(2n-1)\pi}{2W}$, $n = 1, 2, \ldots$
Y22	$\sin(\gamma_n W) = 0$	$\frac{n\pi}{W}$, $n = 0, 1, 2, \ldots$
Y23	$\gamma_n W \tan(\gamma_n W) = h_2 W/k$	
Y31	$\gamma_n W \cot(\gamma_n W) = -h_1 W/k$	
Y32	$\gamma_n W \tan(\gamma_n W) = h_1 W/k$	
Y33	$\tan(\gamma_n W) = [\gamma_n (h_1 + h_2)/k] / [\gamma_n^2 - h_1 h_2 k^{-2}]$	

[a] Index $n = 1, 2, \ldots$ for all cases except Y22 with $n = 0, 1, 2, \ldots$

It is very important to include the $n = 0$ term of the summation only for case Y22. Then Eq. (3) can be written as

$$\sum_{n=0}^{\infty} \frac{Y_n(y') Y_n(y)}{N_y(\gamma_n)} \left\{ \frac{d^2 P_n}{dx^2} - \gamma_n^2 P_n + \delta(x - x') \right\} = 0.$$

This equation is satisfied if the term in brackets is zero for all values of n. That is, function P_n satisfies the following differential equation:

$$\frac{d^2 P_n}{dx^2} - \gamma_n^2 P_n = -\delta(x - x'). \tag{7}$$

Here function P_n has units of meters and parameter γ_n has units of m^{-1}. The solution for P_n may be found using two solutions of the homogeneous equation that satisfy the boundary conditions and are joined appropriately at $x = x'$ (see for example [3, Chapter 1]).

A convenient form of the kernel function may be found with the aid of the reciprocity condition satisfied by all Green's functions: $P_n(x, x') = P_n(x', x)$. From this relation if one of the piecewise-smooth segments of the kernel function is known, the other may be constructed by interchanging the arguments. Using this approach the kernel function may be stated in the form

$$P_n(x,x') = \begin{cases} u(x \mid x') & x > x', \\ u(x' \mid x) & x < x'. \end{cases} \tag{8}$$

Functions $u(x \mid x')$ are listed in Table 3 in the form of exponential functions. Note that all of the arguments of the exponential terms have values which are negative. This form of the kernel function is important for avoiding numerical "overflow" which can occur if the kernel function is expressed with hyperbolic functions. Refer to Appendix A for further discussion of this point.

5.3. Kernel functions for $\gamma_n = 0$

There are nine geometries for which a zero eigenvalue must be included, cases X*IJ*Y22 for $I,J = 1,2,3$. In these cases the kernel function P_0 for cases X*IJ* (not including X22Y22 which is treated later) is found from

$$\frac{d^2 P_0}{dx^2} = -\delta(x - x'). \tag{9}$$

Table 3
Dimensionless functions $u(x \mid x')$ to construct $P_n(x,x')$ for $\gamma_n \neq 0$[a]

Case	$u(x \mid x')$ for $x > x'$. (Use $u(x' \mid x)$ for $x < x'$.) Units: length
X11	$\left(e^{-\gamma(2L-x'-x)} - e^{-\gamma(x-x')}\right)\left(1 - e^{-2\gamma x'}\right) \div [2\gamma(e^{-2\gamma L} - 1)]$
X12	$\left(e^{-\gamma(2L-x'-x)} + e^{-\gamma(x-x')}\right)\left(1 - e^{-2\gamma x'}\right) \div [2\gamma(e^{-2\gamma L} + 1)]$
X13	$\left[\gamma L\left(e^{-\gamma(2L-x'-x)} - e^{-\gamma(2L+x'-x)} + e^{-\gamma(x-x')} - e^{-\gamma(x+x')}\right) + B_2\left(-e^{-\gamma(2L-x'-x)} + e^{-\gamma(2L+x'-x)} + e^{-\gamma(x-x')} - e^{-\gamma(x+x')}\right)\right] \div [2\gamma(\gamma L(1 + e^{-2\gamma L}) + B_2(1 - e^{-2\gamma L}))]$
X21	$\left(e^{-\gamma(x-x')} - e^{-\gamma(2L-x-x')}\right)\left(1 + e^{-2\gamma x'}\right) \div [2\gamma(e^{-2\gamma L} + 1)]$
X22	$\left(e^{-\gamma(x-x')} + e^{-\gamma(2L-x-x')}\right)\left(1 + e^{-2\gamma x'}\right) \div [2\gamma(1 - e^{-2\gamma L})]$
X23	$\left(1 + e^{-2\gamma x'}\right)\left[(\gamma L + B_2)e^{-\gamma(x-x')} + (\gamma L - B_2)e^{-\gamma(2L-x-x')}\right] \div [2\gamma(\gamma L + B_2 + (B_2 - \gamma L)e^{-2\gamma L})]$
X31	$\left(e^{-\gamma(x-x')} - e^{-\gamma(2L-x-x')}\right)\left[(\gamma L - B_1)e^{-2\gamma x'} + \gamma L + B_1\right] \div [2\gamma(\gamma L + B_1 + (\gamma L - B_1)e^{-2\gamma L})]$
X32	$\left(e^{-\gamma(x-x')} + e^{-\gamma(2L-x-x')}\right)\left[(\gamma L - B_1)e^{-2\gamma x'} + \gamma L + B_1\right] \div [2\gamma(\gamma L + B_1 + (B_1 - \gamma L)e^{-2\gamma L})]$
X33	$\{e^{-\gamma(2L-x-x')}(\gamma^2L^2 + \gamma LB_1 - \gamma LB_2 - B_1B_2) + e^{-\gamma(x-x')}(\gamma^2L^2 + \gamma LB_1 + \gamma LB_2 + B_1B_2) + e^{-\gamma(x+x')}(\gamma^2L^2 - \gamma LB_1 + \gamma LB_2 - B_1B_2) + e^{-\gamma(2L-x+x')}(\gamma^2L^2 - \gamma LB_1 - \gamma LB_2 + B_1B_2)\} \div \{2\gamma[\gamma^2L^2 + \gamma LB_1 + \gamma LB_2 + B_1B_2 - (\gamma^2L^2 - \gamma LB_1 - \gamma LB_2 + B_1B_2)e^{-2\gamma L}]\}$

[a] *Note*: $B_1 = h_1L/k$, $B_2 = h_2L/k$.

Table 4
Function $u(x \mid x')$ for case $\gamma = 0$[a]

Case	$u(x \mid x')$ for $x > x'$. (Use $u(x' \mid x)$ for $x < x'$.) Units: length
X11	$x'(1 - x/L)$
X12	x'
X13	$x'[1 - B_2(x/L)/(1 + B_2)]$
X21	$L - x$
X22[b]	$(x'^2 + (x)^2)/(2L) - x + L/3$
X23	$L(1 + 1/B_2 - x/L)$
X31	$(B_1x' - B_1x'x/L + L - x)/(1 + B_1)$
X32	$L(1/B_1 + x'/L)$
X33	$(B_1B_2x' + B_1x' - B_1B_2x'x/L - B_2x + B_2L + L) \div (B_1B_2 + B_1 + B_2)$

[a] *Note*: $B_1 = h_1L/k$, $B_2 = h_2L/k$.
[b] Special temperature solution needed with this pseudo GF.

As with P_n, the kernel functions P_0 for $\gamma_n = 0$ are found by satisfying the homogeneous boundary conditions at $x = 0$ and $x = L$ denoted as X*IJ*. These kernel functions are listed in Table 4 in the form of polynomials. They are identical to the steady 1D GF in Cartesian coordinates (see [7, Appendix X, p. 478]).

5.4. Special case X22Y22

For the case in which all four boundaries of the rectangle are of type 2 (Neumann), the usual GF does not exist and the usual GF solution cannot be used to find the temperature. In this section a pseudo GF is discussed that can be used instead.

In the X22Y22 case the input data to the temperature problem must satisfy a constraint – the sum of the heat passing through the boundaries of the body must be equal to the (negative of the) integral of the heat introduced by volume energy generation. This is equivalent to an energy balance over the volume of the rectangle. If the volume energy generation is zero then the boundary heat fluxes must sum to zero. In addition, the solution for the temperature contains an arbitrary additive constant that must be supplied as the input data.

The pseudo GF, given the name G_{PS}, satisfies the following differential equation:

$$\frac{\partial^2 G_{PS}}{\partial x^2} + \frac{\partial^2 G_{PS}}{\partial y^2} = -\delta(x - x')\delta(y - y') + \frac{1}{LW}. \tag{10}$$

The pseudo GF will be sought in the form given earlier, Eq. (4). Substitute Eq. (4) into Eq. (10) along with the Dirac delta function from Eq. (6) and collect similar terms. Kernel functions $P_n(x,x')$ for $n \neq 0$ satisfy Eq. (7) and are given in Table 3. However, function $P_0(x,x')$ satisfies the following relation

$$\frac{d^2P_0}{dx^2} = -\delta(x-x') + \frac{1}{L}.$$

A solution for P_0 is given by

$$P_0(x,x') = \begin{cases} \left[x^2+(x')^2\right]/(2L) - x' + L/3 & \text{for } x<x', \\ \left[x^2+(x')^2\right]/(2L) - x + L/3 & \text{for } x'<x. \end{cases}$$

The above function is not unique since a different additive constant could be included; the given constant causes the integral of P_0 over $(0, L)$ to be zero.

To find the temperature with the pseudo GF the following integral equation must be used:

$$T(x,y) = \sum_{i=1}^{4} \int_{S_i} \frac{f_i}{k} G_{PS}(x,y \mid x_i', y_i')\, ds_i' + \int\int \frac{g}{k} G_{PS}(x,y \mid x',y')\, dx'\, dy' + \langle T_{X22Y22} \rangle. \tag{11}$$

The sum is over four faces of the rectangle and $\langle T_{X22Y22} \rangle$ is the spatial average temperature in the rectangle.

6. Temperature from spatially uniform effects

The temperature caused by spatially uniform effects is discussed in this section. Exact temperature solutions are of interest for the purpose of high-accuracy verification of finite-element and finite-difference computer codes. Since the problem is linear, a rectangle heated on multiple sides and containing energy generation may be treated as the sum of several problems, each with only one non-homogeneous term. Thus, the temperature caused either by spatially uniform energy generation or spatially uniform boundary effects will be considered one at a time. In the following discussion the non-homogeneous boundary will be considered only at $x=0$ or at $y=0$ without loss of generality because the coordinate system may be rotated or reversed to represent heating on other surfaces.

6.1. Volume energy generation

The temperature caused by volume energy generation is given by the first integral term of Eq. (2). For spatially uniform volume energy generation g_0, the integrals on x' and y' may be carried out independently, tabulated, and used to assemble the temperature series expression for any combination of boundary condition types. The required integrals on $Y_n(y')$ are listed in Table 5 and the integrals of $P_n(x,x')$ are listed in Table 6 for $n \neq 0$ and in Table 7 for $n = 0$. Using these integrals the temperature caused by uniform volume generation g_0 may be written as

$$T(x,y) = \frac{g_0}{k}\int_{x'=0}^{L} P_0(x,x')\, dx' + \frac{g_0}{k}\sum_{n=1}^{\infty}\frac{Y_n(y)}{N_y} \times \int_{y'=0}^{W} Y_n(y')\, dy' \int_{x'=0}^{L} P_n(x,x')\, dx'. \tag{12}$$

Recall that the term containing P_0 is needed only with the Y22 cases. The above expression represents 80 different cases; case X22Y22 requires Eq. (11).

Table 5
Integral of eigenfunction, $\int_{y'=0}^{W} Y_n(y')\, dy'$[a]

Case	$Y_n(y)$	Integral of $Y_n(y')$ over $0<y'<W$
Y11	$\sin(\gamma_n y)$	$W[1-(-1)^n]/(n\pi)$
Y12	$\sin(\gamma_n y)$	$2W \div [(2n-1)\pi]$
Y13	$\sin(\gamma_n y)$	$[1-\cos(\gamma_n W)]/\gamma_n$
Y21	$\cos(\gamma_n y)$	$2W(-1)^{n+1}/[(2n-1)\pi]$
Y22	$\cos(\gamma_n y)$, $\gamma_n \neq 0$ 1, $\gamma_n = 0$	0 for $\gamma_n \neq 0$ W for $\gamma_n = 0$
Y23	$\cos(\gamma_n y)$	$[\sin(\gamma_n W)]/\gamma_n$
Y31	$\sin(\gamma_n(W-y))$	$[1-\cos(\gamma_n W)]/\gamma_n$
Y32	$\cos(\gamma_n(W-y))$	$[\sin(\gamma_n W)]/\gamma_n$
Y33	$\gamma_n W\cos(\gamma_n y) + (h_1W/k)\sin(\gamma_n y)$	$\sin(\gamma_n W) + B_1[1-\cos(\gamma_n W)]/\gamma_n$

[a] *Note*: $B_1 = h_1W/k$.

Table 6
Integral of kernel function, $\int_{x'=0}^{L} P_n(x,x')\, dx'$ for $\gamma \neq 0$

Case	Integral of $P_n(x,x')$ over $0 \leqslant x \leqslant L$, $\gamma \neq 0$. Units: length^2
X11	$\frac{1}{\gamma^2} + \frac{1}{\gamma^2}\left(e^{-\gamma(2L-x)} - e^{-\gamma x} - e^{-\gamma(L-x)} + e^{-\gamma(L+x)}\right) \div (1-e^{-2\gamma L})$
X12	$\frac{1}{\gamma^2} - \frac{1}{\gamma^2}\left(e^{-\gamma(2L-x)} + e^{-\gamma x}\right) \div (1+e^{-2\gamma L})$
X13	$\frac{1}{\gamma^2} + \frac{1}{\gamma^2}\left[B_2\left(e^{-\gamma(2L-x)} + e^{-\gamma(L+x)} - e^{-\gamma(L-x)} - e^{-\gamma x}\right) -\gamma L(e^{-\gamma x} + e^{-\gamma(2L-x)})\right] \div [\gamma L + B_2 + (\gamma L - B_2)e^{-2\gamma L}]$
X21	$\frac{1}{\gamma^2} - \frac{1}{\gamma^2}\left(e^{-\gamma(L-x)} + e^{-\gamma(L+x)}\right) \div (1+e^{-2\gamma L})$
X22	$\frac{1}{\gamma^2}$
X23	$\frac{1}{\gamma^2} - \frac{1}{\gamma^2}B_2\left(e^{-\gamma(L-x)} + e^{-\gamma(L+x)}\right) \div [\gamma L + B_2 + (B_2 - \gamma L)e^{-2\gamma L}]$
X31	$\frac{1}{\gamma^2} + \frac{1}{\gamma^2}\left[B_1\left(e^{-\gamma(L+x)} - e^{-\gamma(L-x)} - e^{-\gamma x} + e^{-\gamma(2L+x)}\right) -\gamma L\left(e^{-\gamma(L-x)} + e^{-\gamma(L+x)}\right)\right] \div [\gamma L + B_1 + (\gamma L - B_1)e^{-2\gamma L}]$
X32	$\frac{1}{\gamma^2} - \frac{1}{\gamma^2}B_1(e^{-\gamma x} + e^{-\gamma(2L-x)}) \div [\gamma L + B_1 + (B_1 - \gamma L)e^{-2\gamma L}]$
X33	$\frac{1}{\gamma^2} + \frac{1}{\gamma^2}\left[(B_1B_2 - \gamma LB_2)e^{-\gamma(L+x)} - (\gamma LB_2 + B_1B_2)e^{-\gamma(L-x)} -(\gamma LB_1 + B_1B_2)e^{-\gamma x} + (B_1B_2 - \gamma LB_1)e^{-\gamma(2L-x)}\right] \div [(\gamma LB_1 + \gamma LB_2 - \gamma^2L^2 - B_1B_2)e^{-2\gamma L} +(\gamma LB_1 + \gamma LB_2 + \gamma^2L^2 + B_1B_2)]$

Table 7
Integral of kernel function with $\gamma = 0$, $\int_{x'=0}^{L} P_0(x,x')\,dx'$

Case	Integral of kernel function $P_0(x,x')$. Units: length^2
X11	$L^2[\frac{1}{2}(x/L) - \frac{1}{2}(x/L)^2]$
X12	$L^2[x/L - \frac{1}{2}(x/L)^2]$
X13	$\frac{1}{2}xL[2 + B_2 - x/L - B_2x/L]/(1 + B_2)$
X21	$\frac{1}{2}L^2[1 - (x/L)^2]$
X22[a]	0
X23	$\frac{1}{2}L^2[B_2 - B_2(x/L)^2 + 2]/B_2$
X31	$\frac{1}{2}L^2[B_1x/L - B_1(x/L)^2 + 1 - (x/L)^2]/(1 + B_1)$
X32	$L^2[x/L - \frac{1}{2}(x/L)^2 + 1/B_1]$
X33	$\frac{1}{2}L^2[B_1B_2(x/L - (x/L)^2) + 2B_1x/L$ $-(B_2 + B_1)(x/L)^2 + B_2 + 2]$ $\div[B_1B_2 + B_1 + B_2]$

[a] Pseudo GF for the X22Y22 case includes an additive constant.

6.2. Uniform boundaries of type 2 or 3

The temperature caused by uniform boundaries of type 2 (heat flux) or type 3 (convection) is given by the second integral term in Eq. (2). For a type 2 boundary, quantity f_i is a uniform heat flux (W/m^2). For a type 3 boundary quantity f_i is usually hT_∞ where h is a heat transfer coefficient and T_∞ is the fluid temperature. The other three boundaries are homogeneous and may be of any type in any combination. If the uniformly heated surface is at $x' = 0$ then the temperature is given by

$$T(x,y) = \frac{f_i}{k} P_0(x,x')|_{x'=0} + \frac{f_i}{k} \sum_{n=1}^{\infty} \frac{Y_n(y)}{N_y} \times \left[\int_{y'=0}^{W} Y_n(y')\,dy' \right] P_n(x,x')|_{x'=0}. \tag{13}$$

The integral falls on the eigenfunction $Y_n(y')$, and the integrals given in Table 5 may be used to construct a series expression for the temperature.

If the heated surface is at $y' = 0$ then the temperature is given by

$$T(x,y) = \frac{1}{W} \frac{f_i}{k} \int_{x'=0}^{L} P_0(x,x')\,dx' + \frac{f_i}{k} \sum_{n=1}^{\infty} \frac{Y_n(y)}{N_y} \times \left[\int_{x'=0}^{L} P_n(x,x')\,dx' \right] Y_n(y')|_{y'=0}. \tag{14}$$

Here the integral falls on the kernel function P_n and the series expression for this temperature may be assembled using functions given in Tables 6 and 7.

An alternative temperature expression may also be constructed from the alternative GF for each of the above cases.

6.3. Uniform temperature boundary (type 1)

The temperature everywhere in the rectangle caused by a boundary uniformly held at temperature $T_0 > 0$ is given by the third integral term in Eq. (2). The recommended form of the temperature is constructed from the GF that has kernel functions parallel to the non-homogeneous boundary.

When the uniform-temperature surface is located at $y' = 0$ then the temperature is given by

$$T(x,y) = T_0 \sum_{n=1}^{\infty} \left[\int_{x'=0}^{L} P_n(x,x')\,dx' \right] \frac{Y_n(y)}{N_y} \frac{dY_n(y')}{dy'}\bigg|_{y'=0}. \tag{15}$$

Here the derivative $-d/dn_i'$ has been replaced by $+d/dy'$ at $y' = 0$. For this expression the integral falls on the kernel function, previously discussed and listed in Table 6. Term P_0 is not present since Y22 is not an option here. The derivative falls on the eigenfunction; expressions for $dY_n(y')/dy'$ are given in Table 8.

The alternate form of the temperature is occasionally needed. The alternate form of the GF is constructed by placing the *eigenfunctions* parallel to the non-zero-temperature boundary (in Eq. (4), interchange x and y and interchange L and W). The alternate temperature for the uniform-temperature surface located at $y' = 0$ is given by

$$T(x,y) = T_0 \frac{dP_0(y,y')}{dy'}\bigg|_{y'=0} + T_0 \sum_{n=1}^{\infty} \frac{X_n(x)}{N_x} \times \int_{x'=0}^{L} X_n(x')\,dx' \frac{dP_n(y,y')}{dy'}\bigg|_{y'=0}, \tag{16}$$

where X_n is the eigenfunction parallel to the x-axis (Table 1). The derivative $-d/dn_i'$ has been replaced by $+d/dy'$ at $y' = 0$.

6.4. Type 3 boundary: second form

The integral expression for the temperature caused by type 3 boundary heating has a second form, developed in this section, which is similar to the type 1 integral expression. On a type 3 boundary, $G = -(k_j/h_j)\,\partial G/\partial n$ (for h_j not zero). Also, on a non-homogeneous type 3 boundary, function f_j may be written as $f_j = h_jT_\infty$ for convection heat transfer to a fluid at T_∞. Replace the above relations for G and f_j into the type 3 integral from Eq. (2) to obtain, for boundary conditions of type 3

$$T(x,y) = -\int_{s_j} T_\infty \frac{\partial G(x,y\,|\,x_j',y_j')}{\partial n_j'}\,ds_j'.$$

This form shows how the temperature caused by a type 3 boundary reduces to the type 1 case when h becomes large. These two representations seem to yield the same

Table 8
Derivative of $Y_n(y')$ with respect to y'

Case	$\mathrm{d}Y_n/\mathrm{d}(y')$	$\mathrm{d}Y_n/\mathrm{d}(y')$ at $y'=0$	$\mathrm{d}Y_n/\mathrm{d}(y')$ at $y'=W$
Y11	$\gamma_n \cos(\gamma_n y')$	$n\pi/W$	$(-1)^n n\pi/W$
Y12	$\gamma_n \cos(\gamma_n y')$	$(2n-1)\pi/(2W)$	0
Y13	$\gamma_n \cos(\gamma_n y')$	γ_n	$\gamma_n \cos(\gamma_n W)$
Y21	$-\gamma_n \sin(\gamma_n y')$	0	$(-1)^n \gamma_n$
Y22	$-\gamma_n \sin(\gamma_n y'),\quad \gamma_n \neq 0$ $0,\quad \gamma_n = 0$	0 for $\gamma_n \neq 0$ 0 for $\gamma_n = 0$	0 for $\gamma_n \neq 0$ 0 for $\gamma_n = 0$
Y23	$-\gamma_n \sin(\gamma_n y')$	0	$-\gamma_n \sin(\gamma_n W)$
Y31	$-\gamma_n \cos[\gamma_n(W-y')]$	$-\gamma_n \cos(\gamma_n W)$	$-\gamma_n$
Y32	$\gamma_n \sin[\gamma_n(W-y')]$	$\gamma_n \sin(\gamma_n W)$	0
Y33	$-\gamma_n^2 W \sin(\gamma_n y') + \frac{h_1 W}{k}\gamma_n \cos(\gamma_n y')$	$\gamma_n(h_1 W/k)$	$-\gamma_n^2 W \sin(\gamma_n W) + \frac{h_1 W}{k}\gamma_n \cos(\gamma_n W)$

infinite series expression for temperature in the few cases that we have investigated. We hope that more definitive conclusions will be reported in the near future.

7. Improved convergence: fully summed series

The convergence speed of a series expression can be greatly enhanced if the series contains the integral of the kernel function, as follows. The series are written in two parts, and the slowly converging part is identified as a 1D steady expression which can be replaced with a fully summed expression. To see this, write the integral of the kernel functions given in Table 6 in the following form:

$$\int_{x'=0}^{L} P_n(x,x')\,\mathrm{d}x' = \frac{1}{\gamma_n^2} + V_n(x,\gamma_n).$$

Here V_n is that part of the integral that contains exponential terms that decay rapidly as n increases. Replace the expression $1/\gamma_n^2 + V_n(x,\gamma_n)$ into the series in place of the above integral wherever it appears. For example, for the temperature caused by specified temperature T_0 at $y=0$ (Eq. (15))

$$T(x,y) = T_0 \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{Y_n(y)}{N_y} \frac{\mathrm{d}Y_n(y')}{\mathrm{d}y'}\bigg|_{y'=0} + T_0 \sum_{n=1}^{\infty} V_n(x,\gamma_n) \times \frac{Y_n(y)}{N_y} \frac{\mathrm{d}Y_n(y')}{\mathrm{d}y'}\bigg|_{y'=0}.$$

The first series which include the term $1/\gamma_n^2$ is identical to the series representation of the 1D temperature distribution that satisfies the appropriate boundary conditions at $y=0$ and $y=W$; refer to Appendix B. This slowly converging series can be replaced by a fully summed form for the 1D, linear-in-y temperature, that is,

$$T(x,y) = T_{\mathrm{1D}}(y) + T_0 \sum_{n=1}^{\infty} V_n(x,\gamma_n) \frac{Y_n(y)}{N_y} \frac{\mathrm{d}Y_n(y')}{\mathrm{d}y'}\bigg|_{y'=0}, \tag{17}$$

where $T_{\mathrm{1D}}(y)$ is the fully summed steady 1D temperature. The second series in the above expression generally converge rapidly and uniformly. Although demonstrated above for heating caused by a type 1 boundary, this technique applies to any temperature expression in which the term $\int P_n(x,x')\,\mathrm{d}x'$ appears, including boundary heating of types 2 and 3. Functions $T_{\mathrm{1D}}(y)$ for boundary-heating cases are listed in Table 9. For temperature caused by uniform internal energy generation, the appropriate fully summed 1D expressions may be taken from Table 7 in the form $T_{\mathrm{1D}}(y)/(g_0 W^2/k)$ by replacing variable x/L by y/W (recall that functions P_0 are the 1D GF). For uniform internal energy generation case X22Y22 is trivial, since the integrals of Y_n and P_0 are both zero.

Table 9
(a) 1D temperature from heating at a type 1 boundary at $y=0$. The other boundary is homogeneous. (b) 1D temperature from heating at a boundary of type 2 ($f=q$) or type 3 ($f=h_1T_\infty$) located at $y=0$. The boundary at $y=W$ is homogeneous

Case	Heated at:	$T_{\mathrm{1D}}(y)/T_0$
Y11	$y=0$	$1-y/W$
Y12	$y=0$	1
Y13	$y=0$	$1-[B_2/(1+B_2)]y/W$
		$T_{\mathrm{1D}}(y)/(fW/k)$
Y21	$y=0$	$1-y/W$
Y22[a]	$y=0$	$\frac{1}{2}(y/W)^2 - y/W + 1/3$
Y23	$y=0$	$1+1/B_2-y/W$
Y31	$y=0$	$(1-y/W)/(1+B_1)$
Y32	$y=0$	$1/B_1$
Y33	$y=0$	$(1-B_2y/W+B_2)/(B_1+B_2+B_1B_2)$

[a] Constructed from the pseudo Green's function.

8. Heat flux

The heat flux is found from the temperature with Fourier's law. There are two heat flux components in the rectangle,

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}.$$

Since taking a derivative of a series degrades the convergence rate, it is important to begin with the best-converging form of the temperature when finding the heat flux. Generally the series for the heat flux are well behaved inside the rectangle, but special care is needed to evaluate the heat flux near the boundaries.

8.1. Type 3 boundary

Heat flux perpendicular to type 3 boundaries should always be evaluated by Newton's law of cooling

$$q = h(T|_{\text{boundary}} - T_\infty)$$

rather than from Fourier's law. The series for boundary temperature in the above expression converge more rapidly than the series for the heat flux computed from the derivative of the temperature.

8.2. Type 2 boundary

At a type 2 boundary, never use a series expression to evaluate the flux perpendicular to the boundary, use the known boundary condition instead. The series expression may give an erroneous value, since eigenfunction $Y_n(y)$ is designed to give zero flux on the boundary.

8.3. Type 1 boundary

The heat flux caused by heating at a type 1 boundary can be troublesome for two reasons. First, the heat flux can be singular near corners between type 1 boundaries if a jump in boundary temperature occurs. Second, even though away from the corners the heat flux is finite, the *series* for heat flux can converge slowly, and even diverge, when the series are evaluated on the boundaries. A general rule for evaluating heat flux at type 1 boundaries is to use the GF with the kernel functions parallel to that boundary.

To calculate the heat flux at a non-homogeneous type 1 boundary (where $T \neq 0$), it is imperative to use the temperature given by Eq. (17). Then the heat flux perpendicular to the heated boundary is given by

$$q_y(x,y) = -k\frac{\mathrm{d}T_{1\mathrm{D}}}{\mathrm{d}y} - kT_0\sum_{n=1}^{\infty} V_n(x,\gamma_n)\frac{1}{N_y} \times \frac{\mathrm{d}Y_n(y)}{\mathrm{d}y}\frac{\mathrm{d}Y_n(y')}{\mathrm{d}y'}\bigg|_{y'=0}. \tag{18}$$

The heat flux in the x-direction based on Eq. (17) is given by

$$q_x(x,y) = -kT_0\sum_{n=1}^{\infty}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{x'=0}^{L} P_n(x,x')\,\mathrm{d}x'\right]\right\} \times \frac{Y_n(y)}{N_y}\frac{\mathrm{d}Y_n(y')}{\mathrm{d}y'}\bigg|_{y'=0}. \tag{19}$$

8.4. Alternate form

In a body heated by a boundary of type 1, the alternate form of the heat flux is needed at or near a side boundary where $T = 0$. Near such a side boundary the convergence of the series for q_y given by Eq. (18) is no longer controlled by rapidly decaying exponential terms and the series may diverge. The alternative form of the heat flux is given by

$$q_y(x,y) = -kT_0\sum_{n=1}^{\infty}\frac{1}{N_y}\frac{\mathrm{d}Y_n(y)}{\mathrm{d}y}\int_{y'=0}^{W} Y_n(y')\,\mathrm{d}y'\frac{\mathrm{d}P_n}{\mathrm{d}x'}\bigg|_{x'=0}.$$

This expression should be used near homogeneous type 1 boundaries $y = 0$ or $y = W$. Away from the boundaries either heat flux relation may be used. To repeat: for heat flux near a type 1 boundary, use kernel functions that are parallel to that boundary.

9. Numerical examples

9.1. Temperature, case X21Y21

Consider the rectangle with heat flux q_0 on boundary $y = 0$, temperature zero at $y = W$, zero heat flux at $x = 0$, and temperature zero at $x = L$. This is case X21Y21. In this example the temperature found from three different series expressions, labeled A, B, and C, will be compared. Using the GF with kernel functions along the x-axis, series A is found from Eq. (14)

$$T_A(x,y) = 2\frac{q_0 W}{k}\sum_{n=1}^{M_A}\cos(\gamma_n y)\left[\frac{1}{W^2\gamma_n^2} + \frac{V_{\mathrm{X21}}(x,\gamma_n,L)}{W^2}\right],$$

where $\gamma_n = (n-1/2)\pi/W$ and $V_{\mathrm{X21}}(x,\gamma_n,L)$ is given in Table 6. Replacing the term $1/\gamma_n^2$ with the linear-in-y 1D temperature gives the best-converging form, series B

$$T_B(x,y) = \frac{q_0 W}{k}\left\{1 - \frac{y}{W} + 2\sum_{n=1}^{M_B}\cos(\gamma_n y)\frac{V_{\mathrm{X21}}(x,\gamma_n,L)}{W^2}\right\}.$$

Series C are found using the alternate GF with kernel functions along the y-axis

$$T_C(x,y) = 2\frac{q_0 W}{k}\sum_{n=1}^{M_C}\cos(\beta_n x)\frac{(-1)^{n+1}}{\beta_n L}\frac{P_n(y'=0,y)}{W},$$

where $\beta_n L = (n-1/2)\pi$ and P_n is given in Table 3 (with $x \to y$, $x' \to y'$, $L \to W$, and $\gamma \to \beta$). The convergence of these series is compared in Table 10 for the case $L/W = 1$. The dimensionless temperature values are listed once and the number of terms to

Table 10
Temperature value and number of terms needed for three different series for the temperature in a rectangle, $L/W = 1$, with specified heat flux q_0 at $y = 0$. Case X21B00Y21B10

x/L	y/W	$T/(q_0 W/k)$	M_A	M_B	M_C
0.01	0.01	0.665314	3550	10	235
0.02	0.01	0.665189	3550	10	225
0.04	0.01	0.664688	3550	10	215
0.08	0.01	0.662680	3550	10	235
0.16	0.01	0.654595	3550	15	245
0.32	0.01	0.621332	3550	15	240
0.64	0.01	0.470723	4150	20	255
0.01	0.08	0.597939	7870	10	45
0.02	0.08	0.597815	7870	10	50
0.04	0.08	0.597320	7870	10	50
0.08	0.08	0.595334	7885	10	50
0.16	0.08	0.587338	7935	10	50
0.32	0.08	0.554460	8170	15	50
0.64	0.08	0.406275	9545	20	50
0.01	0.64	0.188483	17 685	10	15
0.02	0.64	0.188424	17 685	10	15
0.04	0.64	0.188189	17 695	10	15
0.08	0.64	0.187248	17 635	10	15
0.16	0.64	0.183478	17 920	10	15
0.32	0.64	0.168323	18 710	15	15
0.64	0.64	0.107468	23 410	20	15

obtain this temperature is listed for each series. Note that series B are uniformly convergent at various locations in the rectangle, needing no more than 20 terms. At some locations series A require 1000 times more terms than series B. Alternate temperature series C require fewer terms than A but more terms than series B.

9.2. Heat flux, case X11Y11

Consider the rectangle with zero temperature on three faces and the $y = 0$ face at elevated temperature T_0. The normalized heat flux components constructed from Eqs. (18) and (19) are given by

$$\frac{q_x(x,y)}{kT_0/W} = 2\sum_{n=1}^{M} \sin(\gamma_n y) \times \frac{[e^{-\gamma_n(2L-x)} - e^{-\gamma_n(L-x)} + e^{-\gamma_n x} - e^{-\gamma_n(L+x)}]}{(e^{-2\gamma_n L} - 1)}, \tag{20}$$

$$\frac{q_y(x,y)}{kT_0/W} = 1 + 2\sum_{n=1}^{M} \cos(\gamma_n y) \times \frac{[e^{-\gamma_n(2L-x)} - e^{-\gamma_n(L-x)} + e^{-\gamma_n x} - e^{-\gamma_n(L+x)}]}{(e^{-2\gamma_n L} - 1)}, \tag{21}$$

where $\gamma_n = n\pi/W$. These heat flux series converge everywhere inside the rectangle. The convergence is controlled by the rate at which the exponential terms vanish as γ_n increases; the series diverge at $x = 0$ and $x = L$ where one or more exponential terms become unity. At these boundaries the alternative series for the heat flux are needed, constructed from the alternative GF

$$\left.\frac{q_x(x,y)}{kT_0/L}\right|_{\text{alt}} = 2\sum_{n=1}^{M} \cos(\beta_n x)[1 - (-1)^n] \times \frac{[e^{-\beta_n(2W-y)} - e^{-\beta_n y}]}{(1 - e^{-2\beta_n W})} \tag{22}$$

$$\left.\frac{q_y(x,y)}{kT_0/L}\right|_{\text{alt}} = 1 + 2\sum_{n=1}^{M} \sin(\beta_n y) \times \frac{[e^{-\beta_n(2W-y)} + e^{-\beta_n y}]}{(1 - e^{-2\beta_n L})}, \tag{23}$$

where $\beta_n = n\pi/L$. The alternative heat flux expressions are complementary, converging where the previous heat flux expressions do not. In Table 11 the heat fluxes q_x and q_y are given at several locations in the rectangle along with the number of series terms required for the heat flux expressions to converge to the values shown. In this example the heat fluxes at the corners $(0, 0)$ and $(L, 0)$ are singular (there is a jump in temperature).

Calculations were carried out in double precision Fortran 77 on a DEC Alpha computer. Convergence was determined by computing the ratio of the sum of the (absolute values of) the last five terms of the series to the

Table 11
Heat flux and number of terms needed for evaluating series expressions in the rectangle with type 1 boundaries, $L/W = 1$, case X11B00Y11B10

x/L	y/W	$q_xL/(kT_0)$	M (Eq. (20))	M (Eq. (22))	$q_yW/(kT_0)$	M (Eq. (21))	M (Eq. (23))
0.2	0	0.000000	1	[a]	3.411401	35	[a]
0.4	0	0.000000	1	[a]	2.117159	20	[a]
0.6	0	0.000000	1	[a]	2.117159	20	[a]
0.8	0	0.000000	1	[a]	3.411401	35	[a]
0	0.1	−6.257891	[a]	65	0.000000	[a]	5
0.2	0.1	−1.150880	35	70	2.767053	35	65
0.4	0.1	−0.194620	25	75	1.998812	20	70
0.6	0.1	0.194620	25	75	1.998812	20	70
0.8	0.1	1.150880	35	70	2.767053	35	65
1.0	0.1	6.257891	[a]	65	0.000000	[a]	75
0	0.2	−2.972884	[a]	35	0.000000	[a]	5
0.2	0.2	−1.356423	35	40	1.790822	40	35
0.4	0.2	−0.302929	25	40	1.708419	20	35
0.6	0.2	0.302929	25	40	1.708419	20	35
0.8	0.2	1.356423	35	40	1.790822	40	35
1.0	0.2	2.972884	[a]	35	0.000000	[a]	40
0	0.4	−1.214617	[a]	20	0.000000	[a]	5
0.2	0.4	−0.865356	40	20	0.772967	40	20
0.4	0.4	−0.276722	25	20	1.055861	25	20
0.6	0.4	0.276722	25	20	1.055861	25	20
0.8	0.4	0.865356	40	20	0.772967	40	20
1.0	0.4	1.214617	[a]	20	0.000000	[a]	25
0.2	1.0	0.000000	45	1	0.204198	40	15
0.4	1.0	0.000000	25	1	0.329027	20	10
0.6	1.0	0.000000	25	1	0.329027	20	10

[a] Series does not converge at this boundary.

current sum. The series were truncated when this ratio was smaller than 10^{-8}.

10. Summary

In this paper series expressions have been given for steady 2D heat conduction in the rectangle. Single-sum GF provide two forms of the temperature expressions for each geometry depending on the coordinate direction of the kernel functions. Although the two forms are mathematically equivalent, they have complementary numerical convergence properties. The general rule is that the kernel functions should be placed parallel to a non-homogeneous boundary, and then the slowest-converging portions of the series can be fully summed into polynomial expressions. In many cases there is a 1000-fold decrease in the number of series terms needed for evaluation of the temperature. The improved series for the temperature may be differentiated to find convergent expressions for the heat flux, even at specified-temperature boundaries for which the heat flux is notoriously difficult to evaluate. These methods are general in nature and apply to 3D bodies and to bodies in other coordinate systems if the GF may be expressed as Fourier series expansions along one coordinate direction.

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Appendix A. Exponential versus hyperbolic kernel function

This appendix addresses numerical evaluation of the kernel function. The X11Y11 case will be used as an example with the kernel functions in the y-direction.

Two forms of the kernel function will be discussed, the exponential form and the previously published hyperbolic–trigonometric form. The exponential form of the Y11 kernel function is given in Table 3 (for $y < y'$)

$$P_n(y, y') = \frac{(e^{-\gamma_n(2W-y-y')} - e^{-\gamma_n(y'-y)})(1 - e^{-2\gamma_n y})}{2\gamma_n(e^{-2\gamma_n W} - 1)}.$$

The hyperbolic–trigonometric form of this function is given by [12]

$$P_n(y,y') = \frac{\sinh(\gamma_n y)\sinh(\gamma_n(W-y'))}{\gamma_n \sinh(\gamma_n W)}.$$

That these two functions are mathematically equal can be shown with identity $\sinh(z) = (e^z - e^{-z})/2$: however, they have different behaviors when evaluated on a computer.

The hyperbolic–trigonometric form is difficult to evaluate numerically for large n. Although the value of function P_n decreases towards zero as γ_n increases, the value of P_n is determined by a ratio of large numbers. When evaluated on a computer for large values of γ_n, the hyperbolic sine can produce a "numerical overflow" error, indicating a floating-point number whose positive exponent is too large for machine representation.

As a specific example, for case X11Y11 the eigenvalues are given by $\gamma_n = n\pi/L$. For the particular values $y = 0.2$, $y' = 0.5$, $W = 1$, and $L = 3$, when evaluated in single precision Fortran 77 on a DEC Alpha computer, the hyperbolic form of the kernel function can be evaluated only up to $n = 85$. For $n > 85$ numerical overflow occurs and no value is computed. For double precision (with more bits assigned to exponents) the kernel function may be evaluated up to $n = 678$ under the conditions noted above. Since sometimes hundreds or thousands of series terms are needed for evaluating temperature from Green's functions (see Table 10), the hyperbolic-trigonometric form of the kernel function is not adequate for numerical computation.

In contrast, the exponential form of $P_n(y,y')$ may be evaluated numerically for any value of n. First, the denominator is never zero. Second, since every exponential term in the expression has a negative argument, each exponential term is bounded and tends towards zero for large n. When the computer evaluates an exponential term whose argument is large and negative, rather than produce an "underflow" error, the computer returns a zero value. The exponential form of the kernel function is recommended for the numerical evaluation.

Appendix B. Series form of 1D GF

In this appendix the series form of the 1D Cartesian GF is discussed. Temperatures constructed from these GF appear as part of 2D temperature expressions and may be exchanged for numerically better-behaved polynomial forms.

The 1D GF satisfy

$$\frac{d^2G}{dy^2} = -\delta(y-y') \tag{B.1}$$

with boundary conditions

$$k_i\frac{dG}{dn_i} + h_iG = 0, \quad i = 1 \text{ or } 2.$$

The 1D GF in series form are given by

$$G(y,y') = \sum_{n=1}^{\infty}\frac{1}{\gamma_n^2}\frac{Y_n(y)Y_n(y')}{N_y}, \tag{B.2}$$

where eigenfunctions $Y_n(y)$ satisfy Eq. (5) as before. The above expressions apply to cases YIJ for $I,J = 1,2,3$, except for case Y22 which is treated later. Next temperature expressions from these 1D GF will be given.

B.1. Volume energy generation

The 1D steady temperature caused by uniform volume energy generation g_0 is given by

$$T_{1D}(y) = \frac{g_0}{k}\int_{y'=0}^{W} G(y,y')\,dy' = \frac{g_0}{k}\sum_{n=1}^{\infty}\frac{1}{\gamma_n^2}\frac{Y_n(y)}{N_y}\int_{y'=0}^{W} Y_n(y')\,dy'. \tag{B.3}$$

B.2. Heating at boundaries of type 2 or 3

The 1D steady temperature caused by a non-homogeneous boundary of type 2 or 3 at $y = 0$ is given by

$$T_{1D}(y) = \frac{f_j}{k}G(y,y'=0) = \frac{f_j}{k}\sum_{n=1}^{\infty}\frac{1}{\gamma_n^2}\frac{Y_n(y)Y_n(y'=0)}{N_y}. \tag{B.4}$$

Here f_j may be the specified heat flux (type 2 boundary) or the product of the heat transfer coefficient and fluid temperature hT_∞ (type 3 boundary). Heating at surface $y = W$ may be written in a similar fashion.

B.3. Heating at a boundary of type 1

The 1D steady temperature caused by a temperature T_0 at $y = 0$ (type 1 boundary) is given by

$$T_{1D}(y) = T_0\left.\frac{dG(y,y')}{dy'}\right|_{y'=0} = T_0\sum_{n=1}^{\infty}\frac{1}{\gamma_n^2}\frac{Y_n(y)}{N_y}\left.\frac{dY_n(y')}{dy'}\right|_{y'=0}. \tag{B.5}$$

B.4. Polynomial forms

Fully summed polynomial expressions for the above 1D temperature series may be found by using the polynomial form of the 1D GF in place of function $G(y,y')$ in the above expressions. The polynomial form of $G(y,y')$ may be found from the kernel functions P_0 given in Table 4 by making the following change of coordinates: $x \rightarrow y$; $x' \rightarrow y'$; and $L \rightarrow W$. Several temperature expressions in polynomial form are given in Table 9.

B.5. Special case Y22

For the Y22 case (Neumann boundary conditions) Eq. (B.1) cannot be satisfied. However a pseudo Green's function, denoted G_{PS}, can be used that satisfies

$$\frac{d^2 G_{PS}}{dy^2} = -\delta(y - y') + \frac{1}{W} \tag{B.6}$$

with boundary conditions

$$\frac{dG_{PS}}{dn_i} = 0, \quad i = 1 \text{ or } 2.$$

The pseudo GF may be stated in series form given by Eq. (B.2) with appropriate values for $Y_n(y)$, γ_n, and N_y from Tables 1 and 2.

For finding temperature in the 1D Y22 case, the pseudo GF must be applied in a 1D analog to Eq. (11) in which the heating terms satisfy an energy balance and the average temperature appears as an additive constant. Strictly speaking, for case Y22 Eq. (B.4) is not a temperature, since the boundary heat flux represented by f_i is not balanced by another term. However, for the purpose of improving the series convergence for the rectangle, when Eq. (B.4) occurs containing the pseudo GF for case Y22, then the polynomial form given in Table 9(b) may be substituted.

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